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The role of tank-treading motions in the transverse migration of a spheroidal vesicle in a shear flow

Piero Olla

CRS4 via N Sauro 10, 09123 Cagliari, Italy and ISIAAtA-CNR, Università di Lecce, 73100 Lecce, Italy

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Abstract. The behaviour of a spheroidal vesicle, in a plane shear flow bounded from one side by a wall, is analysed when the distance from the wall is much larger than the spheroid radius. It is found that tank-treading motions produce a transverse drift away from the wall, proportional to the spheroid eccentricity and the inverse square of the distance from the wall. This drift is independent of inertia, and is completely determined by the characteristics of the vesicle membrane. The relative strength of the contribution to drift from tank-treading motions and from the presence of inertial corrections, is discussed.

1. Introduction

An important topic in the study of suspensions is that of understanding the ability of shear to induce particle transport perpendicular to the velocity lines [1, 2]. One of the mechanisms responsible for this kind of transport is the presence of lift forces due to inertia, pushing the particles away from the walls [3] (for recent references, see [4]). It is well known that the symmetry properties of the Stokes equation, which governs the dynamics of a purely viscous fluid, do not allow for the presence of lift perpendicular to the flow lines, at least for the case of spherical particles suspended in a plane shear flow [5]. However, mechanisms different from inertia may allow for symmetry breaking, necessary for production of a transverse drift.

A mechanism that is important, if one deals with a suspension of deformable objects, is the ability of shear to induce, under appropriate circumstances, a fixed orientation of the particles in suspension. If the deformable objects are vesicles filled with some other fluid, as in the case of red blood cells, a fixed orientation can be attained when the vesicles are in a state of tank-treading motion [6]. In this state, the membrane and the fluid inside circulate in a steady fashion around the cell interior, while the cell shape and orientation remain constant.

Taking a comoving cartesian system $\{x_1, x_2, x_3\}$, with origin at the cell centre, and oriented in such a way that the unperturbed velocity field has components $\bar{v} = (0, 0, \alpha x_2)$, there will be a velocity perturbation v such that, if the cell is exactly spherical and the inertia is negligible, $v_{1,2}(x_3) = -v_{1,2}(-x_3)$. Under these conditions, a plane wall perpendicular to the x_2 axis will produce a correction v^l to the perturbation field, such that by symmetry $v_2^l = 0$ at the cell centre. An ellipsoidal cell with fixed orientation, instead, will lead to symmetry breaking. Hence, there will be a drift $v^L = v_2^l(0)$, which will depend on the cell eccentricity, on the boundary conditions at the cell surface and, of course, on the distance

from the wall l . Notice, however, that a non-spherical shape is not by itself a sufficient condition for lift. A rigid non-spherical object, for instance, will not keep an orientation fixed in time, carrying on instead a kind of flipping motion [7], whose contribution to lift is going to be much reduced.

Inertia does not play any role in the mechanism for lift outlined above, and it will appear that its effect can be disregarded when the particle is sufficiently close to the wall. Actually, depending on the circumstances, the contribution to drift from tank-treading motions may even turn out to be dominant. The drift of a spherical particle in a bounded shear flow is purely due to inertia; in the case of a particle in a Couette gap with thickness L , $v^L(l) = f(l/L)Re_p\alpha R$, where R is the particle radius, $Re_p = \alpha R^2/\nu$ (with ν the kinematic viscosity of the external fluid) is the particle Reynolds number, and $f(l/L)$ is at most of the order of a few tenths [8]. In the case of a tank-treading motion generated drift, one has instead, in place of Re_p , some parameter describing the non-sphericity of the particle which, as in the case of red cells [9], is not necessarily small.

In this paper, the simplest possible case of a neutrally buoyant, almost spherical vesicle in a purely Newtonian solvent, is taken into consideration. This allows a perturbative analysis around the standard case of a spherical particle. The vesicle is taken at a distance l from the wall such as $R \ll l \ll \sqrt{\nu/\alpha}$; in this range, it is possible at the same time to neglect inertia, and to treat the effect of the wall as a correction to the velocity perturbation due to the vesicle. The technique is similar to the one adopted by Ho and Leal [10] in the case of a spherical particle in a bounded shear flow.

The full problem of the determination of the spheroid shape under the combined effect of the stresses in the solvent, the membrane and the fluid inside is not treated (for references about this problem, see e.g. [11, 12]). Rather, an axisymmetric, ellipsoidal vesicle shape is assumed. Thus, no information on the nature of the membrane and of the fluid inside is utilized. However, two kinds of motions at the vesicle boundary are analysed: one of which is area preserving, mimicking the behaviour of an inextensible membrane, like that of blood cells, and one in which points at the surface move with uniform angular velocity, simulating the case of an immaterial interface.

2. Stokes equation in vector spherical harmonics

In a perturbative analysis around a spherically symmetric situation, it is worthwhile expanding the velocity field in terms of vector spherical harmonics:

$$\begin{aligned} \mathbf{v}(\mathbf{x}) &= \mathbf{v}^s(\mathbf{x}) + \mathbf{v}^e(\mathbf{x}) + \mathbf{v}^m(\mathbf{x}) \\ &= \sum_{lm} [v_{lm}^s(x)\mathbf{x}Y_{lm}(e_x) + v_{lm}^e(x)\nabla Y_{lm}(e_x) + v_{lm}^m(x)[\mathbf{x} \times \nabla]Y_{lm}(e_x)] \end{aligned} \quad (1)$$

where $Y_{lm}(e_x)$ are standard, normalized spherical harmonics and $e_x = \mathbf{x}/x$; the superscripts {s, e and m} stand for scalar, electric and magnetic and come from the origin of this basis as a tool in the study of electromagnetic waves [13].

This basis is clearly orthogonal and the components v_{lm}^{sem} can be obtained in the standard way. Alternatively, one can write

$$v_{lm}^s = \frac{\langle lm|\mathbf{x} \cdot \mathbf{v}\rangle}{x^2} \quad (2)$$

$$v_{lm}^e = -\frac{x^2}{l(l+1)} \langle lm|\nabla_{\perp} \cdot (\mathbf{v} - \mathbf{v}^s)\rangle \quad (3)$$

and

$$v_{lm}^m = \frac{\langle lm | \nabla \cdot [\mathbf{x} \times \mathbf{v}^m] \rangle}{l(l+1)} \quad (4)$$

where the bra-ket notation $\langle lm | f \rangle \equiv f_{lm}(x) = \int d\Omega_x Y_{lm}^*(\mathbf{e}_x) f(\mathbf{x})$, with $d\Omega_x$ the solid angle differential, is used, and ∇_{\perp} is the angular part of the gradient.

At stationarity, an incompressible fluid in creeping flow conditions obeys the time-independent linearized version of the vorticity equation

$$\nabla^2[\nabla \times \mathbf{v}] = 0 \quad (5)$$

together with the continuity equation

$$\nabla \cdot \mathbf{v} = 0. \quad (6)$$

In terms of v_{lm}^{sem} components, the vorticity equation reads

$$\left(\frac{d^2}{dx^2} + \frac{2}{x} \frac{d}{dx} - \frac{l(l+1)}{x^2} \right) f_{lm}^{(1,2)} = 0 \quad (7)$$

where

$$f_{lm}^{(1)} = -v_{lm}^s + \frac{1}{x} v_{lm}^e \quad \text{and} \quad f_{lm}^{(2)} = -v_{lm}^m \quad (8)$$

while the continuity equation takes the form

$$x \frac{dv_{lm}^s}{dx} + 3v_{lm}^s - \frac{l(l+1)}{x^2} v_{lm}^e = 0. \quad (9)$$

From equations (7)–(9), one obtains the ‘outside’ and ‘inside’ solutions:

$$\begin{cases} v_{lm}^s = a_{lm} x^{-1-l} + b_{lm} x^{-3-l} \\ v_{lm}^e = \frac{2-l}{l(l+1)} a_{lm} x^{1-l} - \frac{b_{lm}}{l+1} x^{-1-l} \\ v_{lm}^m = c_{lm} x^{-1-l} \end{cases} \quad (10)$$

and

$$\begin{cases} v_{lm}^s = a'_{lm} x^l + b'_{lm} x^{l-2} \\ v_{lm}^e = \frac{l+3}{l(l+1)} a'_{lm} x^{l+2} - \frac{b'_{lm}}{l} x^l \\ v_{lm}^m = c'_{lm} x^l. \end{cases} \quad (11)$$

The expression for the velocity perturbation by a spherical particle in a strain flow is thus obtained. Remembering the correspondence between fully symmetric, traceless l -tensors and spherical harmonics with given l , the contraction $x^{-2} \alpha_{ij} x_i x_j$, where $\alpha_{ij} = \alpha_{ji}$ and $\alpha_{ii} = 0$ (the convention of summation over repeated indices is adopted), can be expressed as a linear combination of spherical harmonics with $l = 2$. One then finds from equation (10):

$$v_i = \left(\frac{7}{6} a x^{-5} + \frac{5}{3} b x^{-7} \right) x_i \alpha_{jk} x_j x_k - \frac{2}{3} b x^{-5} \alpha_{ij} x_j \quad (12)$$

which is, for $a = -\frac{10b}{7}$ the velocity perturbation due to a sphere of unitary radius put in a strain flow $\bar{v}_i = \frac{2b}{3} \alpha_{ij} x_j$ [14].

3. Velocity perturbation by an ellipsoidal vesicle

3.1. Boundary conditions at the wall

An ellipsoidal cell in a shear flow will feel the effect of the strain component of the flow, which will tend to align strain and ellipsoid axes, while the vortical component will tend to make the cell rotate. If the viscosity of the fluid inside or the rigidity of the membrane are too large, the vesicle will tend to rotate as a rigid body, otherwise it will be in a state of tank-treading motion.

In this configuration the symmetry axis of the ellipsoid will lie in the plane x_2x_3 , at an angle $\bar{\theta}$ with respect to the x_2 , with $\bar{\theta} \rightarrow \pm\pi/4$ for vanishing resistance of the vesicle to deformation, and $\bar{\theta} \rightarrow \pm\pi/2$ at the threshold for the transition to flipping motion [6]. In the two limits, the long side of the ellipsoid tends to be parallel, respectively, to the expanding direction of the strain and to the unperturbed velocity field.

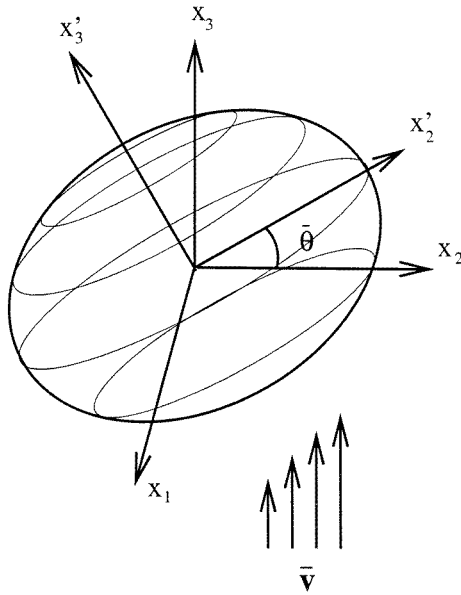


Figure 1. Cell orientation in a plane shear flow $\bar{v} = [0, 0, \alpha x_2]$.

Taking a reference system $\{x'_1, x'_2, x'_3\}$ at the cell centre, as shown in figure 1, with x'_3 along the symmetry axis of the ellipsoid and $x'_1 \equiv x_1$, the equation for the cell surface can be written in the form

$$(1 - \epsilon)(x_1'^2 + x_2'^2) + (1 + \epsilon)x_3'^2 = R^2. \quad (13)$$

For small values of the eccentricity ϵ , the distance from the origin of a point on the surface, with given elevation x'_3 (and corresponding angular coordinates in the unprimed frame θ and ϕ), can be written in the form

$$x(\theta, \phi) \simeq R \left(1 + \frac{\epsilon}{2} \left(1 - 2 \left(\frac{x'_3}{R} \right)^2 \right) \right) = R \left(1 + \frac{\epsilon}{2} \left(1 - 2 \frac{(\cos \bar{\theta} x_3 - \sin \bar{\theta} x_2)^2}{R^2} \right) \right). \quad (14)$$

In a state of tank-treading motion, the membrane is assumed to move following two possible laws. In the first case, for each value of x_1 , points on the surface are taken to move at

constant speed:

$$v_B = \frac{1}{2} B \alpha R(x_1) \quad \text{with } R(x_1)^2 = R^2 - (1 - \epsilon)x_1^2 \quad (15)$$

remaining on the elliptic trajectory, which corresponds to the given section at constant x_1 of the cell surface. Such motion is locally area preserving, and approximates the behaviour of an inextensible membrane, with negligible resistance to bending and shear. For small ϵ , one expects $B \simeq 1 + \beta\epsilon$; thus, for $\epsilon = 0$, the expression for v_B reduces to that for a sphere immersed in the given shear flow. In terms of components

$$\mathbf{v}_B = v_B \left(0, \frac{x'_3}{\sqrt{x'^2_3 + (\gamma x'_2)^2}}, \frac{x'_2}{\sqrt{x'^2_2 + (x'_3/\gamma)^2}} \right) \quad (16)$$

where $\gamma = \sqrt{\frac{1-\epsilon}{1+\epsilon}}$. For small ϵ :

$$\mathbf{v}_B \simeq \frac{\alpha}{2} \left[0, -x'_3 \left(1 + \epsilon \left(\beta + \frac{x'^2_1}{2x'^2_\perp} + \frac{2x'^2_2}{x'^2_\perp} \right) \right), x'_2 \left(1 + \epsilon \left(\beta + \frac{x'^2_1}{2x'^2_\perp} - \frac{2x'^2_3}{x'^2_\perp} \right) \right) \right] \quad (17)$$

where $x'^2_\perp = x'^2_2 + x'^2_3$. Using equations (2) and (17), one finds

$$\mathbf{v}_B^s \simeq -\epsilon \alpha \mathbf{x} \frac{x'_2 x'_3}{x^2} \quad (18)$$

and

$$\mathbf{v}_B^e \simeq \epsilon \alpha \left(\frac{\mathbf{x}}{x^2} - \frac{\mathbf{x}_\perp}{x'^2_\perp} \right) x'_2 x'_3 \quad (19)$$

It will appear that only the $l = 2$ part of \mathbf{v}_B must be taken into consideration in order to calculate the lift. Using equation (3), a simple calculation leads to the result

$$\mathbf{v}_B = \mathbf{v}_B^m - \frac{5}{6} \epsilon \alpha \mathbf{x} \frac{x'_2 x'_3}{x^2} - \frac{\epsilon \alpha}{12} [0, x'_3, x'_2] + \text{HH} \quad (20)$$

where HH indicates higher harmonics. In terms of unprimed variables and components

$$\mathbf{v}_B = \mathbf{v}_B^m - \frac{5}{12} \epsilon \alpha \mathbf{x} \frac{(x^2_3 - x^2_2) \sin 2\bar{\theta} + 2x_2 x_3 \cos 2\bar{\theta}}{x^2} - \frac{\epsilon \alpha}{12} [0, -x_2 \sin 2\bar{\theta} + x_3 \cos 2\bar{\theta}, x_3 \sin 2\bar{\theta} + x_2 \cos 2\bar{\theta}] + \text{HH}. \quad (21)$$

The second membrane behaviour that is considered, is characterized by a constant angular velocity motion:

$$\mathbf{v}_B = \frac{B \alpha}{2} [0, -x'_3/\gamma, \gamma x'_2] \simeq \frac{\alpha}{2} (1 + \beta\epsilon) [0, -x'_3, x'_2] - \frac{\alpha \epsilon}{2} [0, x'_3, x'_2]. \quad (22)$$

In terms of unprimed variables

$$\mathbf{v}_B \simeq \frac{\alpha}{2} (1 + \beta\epsilon) [0, -x_3, x_2] - \frac{\alpha \epsilon}{2} [0, -x_2 \sin 2\bar{\theta} + x_3 \cos 2\bar{\theta}, x_3 \sin 2\bar{\theta} + x_2 \cos 2\bar{\theta}]. \quad (23)$$

The behaviour described in equations (22) and (23) should mimic the motion of a droplet; however, this is also the approximation for an area preserving membrane motion, used by Keller and Skalak in [6]. Thus, comparing the velocity perturbation and the lift produced by the two membrane motions, prescribed in equations (17) and (23), gives, among other things, an idea on the acceptability of such an approximation, also in view of the simplifications that become possible in the large ϵ regime.

3.2. Calculation of the perturbation in an infinite fluid

If the membrane velocity $\mathbf{v}_B(\theta, \phi)$ and the cell orientation $\bar{\theta}$, produced by the external flow $\bar{\mathbf{v}}(\theta, \phi, x)$, are known, the velocity perturbation \mathbf{v} can be obtained from the boundary condition at the vesicle surface $x = x(\theta, \phi)$:

$$\mathbf{v}_B(\theta, \phi) = \bar{\mathbf{v}}(\theta, \phi, x(\theta, \phi)) + \mathbf{v}(\theta, \phi, x(\theta, \phi)). \quad (24)$$

For small values of ϵ , it is possible to solve equation (24) perturbatively to $\mathcal{O}(\epsilon)$:

$$\mathbf{v}_B^{(0)}(\theta, \phi) = \bar{\mathbf{v}}(\theta, \phi, R) + \mathbf{v}^{(0)}(\theta, \phi, R) \quad (25a)$$

$$\mathbf{v}_B^{(1)}(\theta, \phi) = \bar{\mathbf{v}}(\theta, \phi, x(\theta, \phi)) + \mathbf{v}^{(0)}(\theta, \phi, x(\theta, \phi)) + \mathbf{v}^{(1)}(\theta, \phi, R) \quad (25b)$$

where, for $x(\theta, \phi)$, the $\mathcal{O}(\epsilon)$ correct expression, provided by equation (14) can be used. In equation (25a) $\mathbf{v}_B^{(0)}$ is the velocity at the surface of the sphere, with $x^{(0)}(\theta, \phi) = R$, immersed in the same flow:

$$\mathbf{v}_B^{(0)}(\theta, \phi) = \frac{\alpha}{2}[0, -\mathbf{x}_3^{(0)}, \mathbf{x}_2^{(0)}]. \quad (26)$$

In this way, $\mathbf{v}^{(0)}$ is the velocity perturbation by a sphere in a shear flow, given by equation (12):

$$\mathbf{v}^{(0)} = \frac{5}{2} \left(\frac{R^4}{x^4} - \frac{R^2}{x^2} \right) \frac{\alpha x_2 x_3 \mathbf{x}}{x^2} - \frac{\alpha R^4}{x^4} [0, x_3, x_2]. \quad (27)$$

Expanding equation (25b) in vector spherical harmonics and using equation (10) then allows us to calculate $\mathbf{v}^{(1)}$ for $x > x(\theta, \phi)$.

For the first membrane motion, described by equation (21), one has

$$\mathbf{v}_B^{(1)} = \frac{\alpha}{2} \left(\left(\beta + \frac{x_1^2}{2x_\perp^2} \right) \epsilon + \frac{\delta x}{R} \right) [0, -x_3, x_2] + \mathbf{v}_C \quad (28)$$

where

$$\mathbf{v}_C = -\frac{5}{12} \epsilon \alpha \mathbf{x} \frac{(x_3^2 - x_2^2) \sin 2\bar{\theta} + 2x_2 x_3 \cos 2\bar{\theta}}{R^2} - \frac{\epsilon \alpha}{12} [0, -x_2 \sin 2\bar{\theta} + x_3 \cos 2\bar{\theta}, x_3 \sin 2\bar{\theta} + x_2 \cos 2\bar{\theta}] + \mathbf{H}\mathbf{H} \quad (29)$$

and

$$\delta x = \frac{R\epsilon}{2} \left(1 - 2 \frac{(\cos \bar{\theta} x_3 - \sin \bar{\theta} x_2)^2}{R^2} \right). \quad (29a)$$

(The identification $\mathbf{x}^{(0)} = \mathbf{x}$ is allowed at the order in ϵ considered.) It appears (details are given in the appendix) that the condition on the absence of external torque on the vesicle corresponds to the $l = 1$ components of \mathbf{v}^m , being identically zero. This is achieved by choosing appropriately the coefficient β , and consequently the value of the tank treading velocity. The present analysis is directed to calculating the lift at a large distance from the wall. Hence, from equation (10), the leading contribution is from the $\mathbf{v}_{l=2}^s$ terms. Substituting equations (14), (27) and (28) into equation (25b), keeping only $\mathcal{O}(\epsilon)$ terms, and neglecting \mathbf{v}^m and $l > 2$ components, one obtains for $\mathbf{v}^{(1)}$

$$\mathbf{v}_{x=R}^{(1)} = \mathbf{v}_C + \frac{5\alpha\delta x}{R} \left(\frac{x_2 x_3 \mathbf{x}}{R^2} - \frac{1}{2} [0, x_3, x_2] \right). \quad (30)$$

Using equations (29) and (29a) together with equations (2) and (3), one finds:

$$\mathbf{v}_{x=R}^{(1)s} = -\frac{\epsilon\alpha\mathbf{x}}{2R^2} [(x_3^2 - x_2^2) \sin 2\bar{\theta} + 2x_2 x_3 \cos 2\bar{\theta}] \quad (31)$$

and:

$$\mathbf{v}_{x=R}^{(1)e} \simeq -\epsilon\alpha R^2 \nabla \frac{(x_3^2 - x_2^2) \sin 2\bar{\theta} + (2 \cos 2\bar{\theta} + 30)x_2 x_3}{x^2} - \frac{5\epsilon\alpha}{R^2} (\cos \bar{\theta} x_3 - \sin \bar{\theta} x_2)^2 \left(\frac{x_2 x_3 \mathbf{x}}{R^2} - \frac{1}{2} [0, x_3, x_2] \right). \quad (32)$$

All terms in equations (31) and (32) are clearly $l = 2$ with the exception of the last line of equation (32), which is a combination of \mathbf{v}^e and \mathbf{v}^m terms. After some tedious calculations, expedited by the use of equation (3), the following expression for the $l = 2$ part of $\mathbf{v}_{x=R}^{(1)e}$ is obtained:

$$\mathbf{v}_{x=R}^{(1)e} = \epsilon\alpha R^2 \nabla \left(\frac{\sin 2\bar{\theta}}{x^2} \left(\frac{5x_1^2}{42} - \frac{x_2^2}{56} - \frac{17x_3^2}{168} \right) - \left(\frac{25}{84} + \frac{\cos 2\bar{\theta}}{12} \right) \frac{x_2 x_3}{x^2} \right). \quad (33)$$

The leading term at large distance is, in the notation of equation (10), the contribution $a_{2m}x^{-3}$ to \mathbf{v}^s . From equations (31) and (33), using equation (10), one therefore obtains the large distance result:

$$\mathbf{v}^{(1)} \simeq \frac{\epsilon\alpha R^3 \mathbf{x}}{x^3} \left(\frac{\sin 2\bar{\theta}}{x^2} \left(\frac{5x_1^2}{14} + \frac{25x_2^2}{56} - \frac{45x_3^2}{56} \right) - \left(\frac{25}{28} + \frac{5 \cos 2\bar{\theta}}{4} \right) \frac{x_2 x_3}{x^2} \right). \quad (34)$$

The same identical calculations can be carried on for the second kind of membrane motion described by equation (23), with the final result:

$$\mathbf{v}^{(1)} \simeq \frac{\epsilon\alpha R^3 \mathbf{x}}{x^3} \left(\frac{\sin 2\bar{\theta}}{x^2} \left(\frac{5x_1^2}{14} + \frac{15x_2^2}{14} - \frac{10x_3^2}{7} \right) - \left(\frac{25}{28} + 5 \cos 2\bar{\theta} \right) \frac{x_2 x_3}{x^2} \right). \quad (35)$$

One question that comes naturally at this point is whether one could extend the analysis to a non-perturbative regime, expanding directly equation (24) into vector spherical harmonics and then, after imposing a cut-off on l and m , solving numerically the resulting linear system in the unknowns a_{lm} , b_{lm} and c_{lm} . This could be useful in analysing the behaviour of vesicles of a general, strongly non-spherical shape, but it is not the most efficient way of proceeding, because the matrix associated with the system is strongly ill-conditioned. Of course, in the case of an ellipsoid immersed in a shear flow, undergoing a linear tank-treading motion like the one of equation (22), the theory of Keller and Skalak [6] can be utilized to obtain analytic expressions for the velocity field. In the more general case, it turns out that it is possible to modify the initial vector spherical harmonics basis, in such a way that the velocity field is obtained directly, without having to invert any ill-conditioned matrices [15].

4. Calculation of lift

The calculations in the previous section referred to an unbounded flow situation. A wall at a large distance from the vesicle will cause a correction \mathbf{v}^l to the velocity perturbation \mathbf{v} obtained in the previous section. This can be calculated using the boundary condition provided by making the correction at the wall equal to minus the velocity given by equations (34) or (35). The higher-order corrections are then obtained, using the value, alternately on the vesicle surface and on the wall, of the previous calculated correction as a boundary condition, in a series of images and counter-images, analogous to those of electrostatics. If the distance from the wall is large, it is possible, however, to stop at the first image.

The technique at this point is standard [10, 16]; one introduces scalar and vector potential ϕ and \mathbf{A} , such as

$$\mathbf{v}^l = \nabla\phi + \nabla \times \mathbf{A} \quad (36)$$

where

$$\nabla^2\phi = 0 \quad \text{and} \quad \nabla \cdot \mathbf{A} = 0. \quad (37)$$

The first of equation (37) is a consequence of incompressibility, while the second is a gauge condition. From here, the vorticity equation takes the form of a biquadratic:

$$\nabla^2\nabla^2\mathbf{A} = 0. \quad (38)$$

Taking the wall at $x_2 = l$ positive, parallel to the $x_{1,3}$ axes, it is useful to Fourier transform all quantities with respect to x_1 and x_3 :

$$f(\mathbf{x}) = \int \frac{dk_1 dk_3}{(2\pi)^2} e^{i(k_1x_1+k_3x_3)} \tilde{f}(k_1, k_3, x_2).$$

The gauge condition on \mathbf{A} takes then the form

$$\tilde{A}_3 = -\frac{k_1}{k_3}\tilde{A}_1 + \frac{i}{k_3}\tilde{A}'_2 \quad (39)$$

where the prime indicates derivative with respect to x_2 . The gauge is definitely fixed by requiring that \mathbf{A} does not contain any potential contribution \mathbf{a} such as $\nabla^2\mathbf{a} = 0$. With this condition, the solution of equation (38) gives

$$\tilde{\mathbf{A}} = \hat{\mathbf{A}}(k_1, k_3)(x_2 - l) \exp(k(x_2 - l)) \quad (40)$$

where $k = \sqrt{k_1^2 + k_3^2}$. The first of equations (37), instead, gives for ϕ :

$$\tilde{\phi} = \hat{\phi}(k_1, k_3) \exp(k(x_2 - l)). \quad (41)$$

Using equations (36) and (39), the expression for the velocity correction becomes, in terms of Fourier components

$$\begin{cases} \tilde{v}_1^l = -\frac{k_1}{k_3}\tilde{A}'_1 + \frac{i}{k_3}\tilde{A}''_2 - ik_3\tilde{A}_2 + ik_1\tilde{\phi} \\ \tilde{v}_2^l = \frac{ik^2}{k_3}\tilde{A}_1 + \frac{k_1}{k_3}\tilde{A}'_2 + \tilde{\phi}' \\ \tilde{v}_3^l = ik_1\tilde{A}_2 - \tilde{A}'_1 + ik_3\tilde{\phi} \end{cases} \quad (42)$$

and imposing the boundary condition $\tilde{v}^l(k_1, k_3, l) = -\tilde{v}(k_1, k_3, l)$, one finds

$$\begin{cases} \hat{v}_1 = \frac{k_1}{k_3}\hat{A}_1 - \frac{2ik}{k_3}\hat{A}_2 - ik_1\hat{\phi} \\ \hat{v}_2 = -\frac{k_1}{k_3}\hat{A}_2 - k\hat{\phi} \\ \hat{v}_3 = \hat{A}_1 - ik_3\hat{\phi} \end{cases} \quad (43)$$

where $\hat{v}(k_1, k_3) = \tilde{v}(k_1, k_3, l)$. Solution of this system gives:

$$\begin{cases} \hat{\phi} = \frac{i[-k_1k_3\hat{v}_1 + 2ik_3k\hat{v}_2 - k_1^2\hat{v}_3]}{2k_3k^2} \\ \hat{A}_1 = \frac{k_1k_3\hat{v}_1 - 2ik_3k\hat{v}_2 - (k^2 + k_3^2)\hat{v}_3}{2k^2} \\ \hat{A}_2 = \frac{i(k_3\hat{v}_1 - k_1\hat{v}_3)}{2k}. \end{cases} \quad (44)$$

At this point, one finds \bar{v}_2^1 , by substitution of equation (44) into (42); for $x_2 = 0$

$$\bar{v}_2^1(k_1, k_3, 0) = -[ik_1 l \hat{v}_1 + (1 + kl) \hat{v}_2 + ik_3 l \hat{v}_3] \exp(-kl). \quad (45)$$

The lift is found from inverse Fourier transform at $x_1 = x_3 = 0$:

$$v^L(l) = \int \frac{d^2k}{(2\pi)^2} v_2^1(k_1, k_3, 0).$$

Of all the contributions to v entering equation (45), only those proportional to $\sin 2\bar{\theta}$ in $v^{(1)}$ give a non-zero result. Writing this in an explicit way:

$$v^L(l) = -\frac{1}{(2\pi)^2} \int_0^\infty k dk \int_0^\infty \hat{x} d\hat{x} \int_0^{2\pi} d\varphi \int_0^{2\pi} d\psi \exp(-ik\hat{x} \cos \psi - kl) \\ \times [klv_1 \cos \varphi + (1 + kl)v_2 + iklv_3 \sin \varphi] \quad (46)$$

where $\hat{x} = \sqrt{x_1^2 + x_3^2}$, $x_1 = \hat{x} \cos \theta$, $k_1 = k \cos \varphi$ and $\psi = \theta - \varphi$. In equation (46), one can write, from equations (34) and (35):

$$v = \frac{\epsilon \alpha R^3 \sin 2\bar{\theta}}{\hat{x}^5} (2Cl^2 + 2(D - C)\hat{x}^2 \cos 2\bar{\theta} - D\hat{x}^2)[\hat{x} \cos \theta, l, \hat{x} \sin \theta] \quad (47)$$

where all terms giving zero contribution in the integrals of equation (46) are disregarded. The integrals in equation (46) can be carried out analytically, exploiting the following properties of the Bessel functions $J_\nu(x)$ [17]:

$$\int_0^{2\pi} d\psi \exp(-i\alpha \cos \psi) = 2\pi J_0(\alpha) \quad (48)$$

$$\int_0^{2\pi} d\psi \cos \psi \exp(-i\alpha \cos \psi) = 2\pi i J_1(\alpha) \quad (49)$$

and

$$\int_0^\infty x^\mu dx J_\nu(\beta x) \exp(-\alpha x) = (-1)^\mu \frac{d^\mu}{d\alpha^\mu} \frac{(\sqrt{\alpha^2 + \beta^2} - \alpha)^\nu}{\beta^\nu \sqrt{\alpha^2 + \beta^2}}. \quad (50)$$

The final result is

$$v^L = -\frac{\epsilon \alpha C R^3 \sin 2\bar{\theta}}{l^2}. \quad (51)$$

Thus, the magnitude and the direction of the lift depend on the coefficient of the x_2^2 terms in the expression for $v^{(1)}$ given by equations (34) and (35). It is known [6], and a simple argument for it is given in the appendix, that for sufficiently small values of ϵ , $\bar{\theta} = \pm \frac{\pi}{4}$, with the plus sign when the ellipsoid is oblate ($\epsilon > 0$), and the minus when this is prolate ($\epsilon < 0$). This leads then, both in the inextensible membrane case of equation (34), and in the 'droplet' case of equation (35), to a drift away from the wall, which is directly proportional to the inverse square of the distance from the wall, and to the eccentricity ϵ :

$$v^L = -\frac{C|\epsilon|\alpha R^3}{l^2}. \quad (52)$$

The values of the constant C are obtained from equations (34) and (35); one finds in the two cases, respectively, $C = \frac{25}{112}$ and $C = \frac{15}{28}$.

5. Conclusions

The main result that has been obtained in this paper is that tank-treading motions are able to produce a transverse drift of vesicles in a sheared suspension. This effect is mainly localized near the walls, where it can dominate that of the inertial corrections, on which current theories on the lift of particles in suspension are based. It is possible to estimate the thickness of the region where this happens. Using the expression for the inertial drift of a spherical particle, in a bounded shear flow with $0 < l < L$: $f(l/L) \sim f^0(l/L - 0.5)$, where f^0 is a constant of the order of a few tenths [8], one finds from equation (51) that the drift from tank treading motions remains dominant as long as

$$\frac{l}{R} < \left(\frac{C\epsilon L}{f^0 Re_p R} \right)^{\frac{1}{3}}. \quad (53)$$

Thus, thanks to the smallness in most situations of the particle Reynolds number Re_p , the width of the region where the drift from tank-treading motions is dominant can be so large, that both approximations of large l/R and small ϵ , used in this paper, can be satisfied at the same time.

However, the most interesting situation is that of strongly non-spherical vesicles, for which $\epsilon = \mathcal{O}(1)$. It is known that, for large values of ϵ , the vesicle tends to have its major axis aligned with the unperturbed flow (which corresponds to $\sin 2\hat{\theta} = 0$), and then to make the transition to a flipping motion [6]. Thus, there must be some critical $\bar{\epsilon}$ for which, with the other parameters fixed, the drift velocity achieves its maximum value.

In the present theory, due to the smallness of ϵ , one has: $\hat{\theta} \simeq \pm\pi/4$, and the effect of destruction of tank-treading motions, produced by ϵ becoming too large, is not accounted for. At the same time, the perturbative calculation leading to the large-distance behaviour of the velocity disturbance v , becomes unreliable. However, extrapolating equation (53) to large values of ϵ , the effect of tank-treading motions (when present) is expected to become dominant in all situations of suspensions flowing in narrow gaps. Thus a non-perturbative extension of the theory at large ϵ would be advisable.

Red cells are a physical system in which such a non-perturbative theory could be applied. However, experimental observations [18, 19], as well as theoretical analysis [6], indicate that, due to their membrane viscosity and that of the haemoglobin inside, these cells undergo tank treading motions only when immersed in very viscous solvents, or in the presence of very strong shear stresses. It is clear on the other hand, that inertia cannot by itself account for phenomena like the concentration of red cells near the axis of small blood vessels: the so-called Fahraeus–Lindqvist effect [20]. Using typical parameters for a red cell and a small vessel [9], $R \sim 4\mu$, $L \sim 100\mu$, $\alpha \sim 200 \text{ s}^{-1}$ (i.e. $\bar{v} \sim 1 \text{ cm s}^{-1}$), one finds that a transverse drift of the order of L , would occur only after the cell has travelled already for several centimetres, much more than the length of a typical small vessel. Thus, a fixed orientation and strong departure from sphericity remain the necessary ingredients for a sufficiently large lift, and some different mechanism for maintaining a fixed orientation in the cell must be found.

The most important limitation of an analysis like the one in this paper is the fact that the membrane motion and the cell shape are imposed from the outside. This, among other things, makes it impossible to analyse the behaviour of the cell, when the distance from the wall becomes comparable to the radius R . At the same time, there remain open several questions on which kind of membrane motions and cell shapes are possible, for instance whether there could be membranes, whose response to an external shear field leads to a drift towards the nearest wall, instead of away from it. The two behaviours considered in this work lead to lifts of different magnitudes, but both directed away from the wall.

The fact that different membrane behaviours lead to the cell moving in one direction or another suggests the possibility of controlling the cell motion through the membrane stiffness and the internal fluid viscosity. This could be of great importance in microsurgery, where new methods to deliver drugs using microcapsules that break and drop their contents when they are on the selected target, are taken into consideration. Already now, it seems realistic to have cells whose stiffness varies with temperature and whose tendency to stay away from the walls could be controlled from the outside. Carefully designing the cell structure (and its response to external stresses or other stimuli), could allow in principle, a fuller control on the cell trajectory, without having to rely on internal motors and sources of energy.

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Appendix. Torque and energy balance

In a steady state of tank-treading motion, a vesicle in a shear flow will feel no external torque, while the external work will be equal to the energy dissipated in the membrane and the cell interior. In this paper, the membrane is supposed to oppose no resistance to bending and to shear, and consequently, to be dissipation free. The simultaneous satisfaction of these two conditions fixes the magnitude of the tank-treading velocity v_B and the orientation angle $\bar{\theta}$.

The external torque is calculated from the value of the stress tensor T_{ij} at a spherical surface enclosing the vesicle:

$$T_{ij} = \mu v (\partial_i v_j + \partial_j v_i) \quad (\text{A1})$$

where μ is the fluid density. The force and the torque on an infinitesimal element dS_i of the cell surface, oriented towards the outside, will be equal to, respectively,

$$d\Pi_i = T_{ij} dS_j \quad \text{and} \quad M_i = \epsilon_{ijk} x_j d\Pi_k. \quad (\text{A2})$$

From here, the total torque acting of the spherical surface in exam, can be expressed, after a few manipulations, in the form,

$$\mathbf{M} = \mu v x \int d\Omega_x \{ [\mathbf{x} \times \nabla] (\mathbf{x} \cdot \mathbf{v}) + [\mathbf{x} \times (\mathbf{x} \cdot \nabla) \mathbf{v}] - [\mathbf{x} \times \mathbf{v}] \}. \quad (\text{A3})$$

The decomposition in vector spherical harmonics is used again:

$$\mathbf{v} = v^s \mathbf{x} Y + v^e \nabla Y + v^m [\mathbf{x} \times \nabla] Y \quad (\text{A4})$$

where the notation $v^{\text{sem}} Y \equiv \sum_{lm} v_{lm}^{\text{sem}} Y_{lm}$ is used.

Substituting back into equation (A3), one obtains the result that only the v^m components contribute to \mathbf{M} :

$$\mathbf{M} = -\mu v x^5 (v^m/x)' \int d\Omega_x \nabla Y \quad (\text{A5})$$

and, remembering the correspondence between spherical harmonics and irreducible tensors, it appears that only the $l = 1$ components survive. The no-torque condition takes then the form, from equation (10),

$$c_{l=1} = 0. \quad (\text{A6})$$

Turning to the requirement of energy conservation and steady state, the equation for the balance between dissipation and external work can be written in the form

$$W = \int v_B \cdot d(\mathbf{\Pi}' - \mathbf{\Pi}) = 0 \quad (\text{A7})$$

where W is the membrane dissipation, taken equal to zero, and $d\mathbf{\Pi}'$ is the force with which the fluid inside the cell acts on the infinitesimal element dS of the membrane. If the vesicle is perfectly spherical, it will rotate as a whole, so that $W = 0$, $d\mathbf{\Pi}' = 0$ and equation (A7) will coincide with the no-torque condition $M = 0$. In fact, in this case only one parameter, v_B , remains to be calculated.

For small non-zero ϵ , W is a linear functional of the external velocity field and can be decomposed into a contribution due to the strain S and another to the vorticity ω of \bar{v} :

$$W = W^S(\bar{\theta}, \epsilon, S) + W^\omega(\bar{\theta}, \epsilon, \omega). \quad (\text{A8})$$

If $\omega = 0$ the cell will tend to align with the strain axes of \bar{v} , i.e.: $\bar{\theta} = \pm \frac{\pi}{4}$; therefore

$$W^S\left(\pm \frac{\pi}{4}, \epsilon, S\right) = W^S(\bar{\theta}, 0, S) = 0 \quad (\text{A9})$$

and the first non-zero derivative of W^S is $\partial_{\bar{\theta}} \partial_\epsilon W^S(\pm \frac{\pi}{4}, 0, S)$. Turning to the vortical component, writing

$$M = \int dS A_M \quad \text{and} \quad W^\omega = \int dS A_W \quad (\text{A10})$$

one observes that

$$A_W = (c + \mathcal{O}(\epsilon))A_M + \mathcal{O}(\epsilon^2). \quad (\text{A11})$$

Thus, if the no-torque condition $M = 0$ is satisfied, one has:

$$W^\omega(\bar{\theta}, \epsilon, \omega) = \frac{\epsilon^2}{2} \partial_\epsilon^2 W^\omega(\bar{\theta}, 0, \omega) + \mathcal{O}(\epsilon^3)$$

and one finds from equation (A7)

$$\bar{\theta} = \pm \frac{\pi}{4} - \frac{\epsilon}{2} \frac{\partial_\epsilon^2 W^\omega}{\partial_{\bar{\theta}} \partial_\epsilon W^S} + \mathcal{O}(\epsilon^2). \quad (\text{A12})$$

Hence, almost spherical vesicles tend to carry on a tank-treading motion, with the symmetry axis aligned with one of the strain axes of the external flow.

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